COMPLEX NUMBERS AND QUADRATIC EQUATIONS

❖ *Mathematics is the Queen of Sciences and Arithmetic is the Queen of Mathematics. – GAUSS* ❖

4.1 Introduction

In earlier classes, we have studied linear equations in one and two variables and quadratic equations in one variable. We have seen that the equation $x^2 + 1 = 0$ has no real solution as $x^2 + 1 = 0$ gives $x^2 = -1$ and square of every real number is non-negative. So, we need to extend the real number system to a larger system so that we can find the solution of the equation $x^2 = -1$. In fact, the main objective is to solve the equation $ax^2 + bx + c = 0$, where $D = b^2 - 4ac < 0$, which is not possible in the system of real numbers.

W. R. Hamilton
(1805-1865)

4.2 Complex Numbers

Let us denote $\sqrt{-1}$ by the symbol i. Then, we have $i^2 = -1$. This means that i is a solution of the equation $x^2 + 1 = 0$.

A number of the form $a + ib$, where a and b are real numbers, is defined to be a

complex number. For example, $2 + i3$, $(-1) + i\sqrt{3}$, $4 + i\left(\dfrac{-1}{11}\right)$ are complex numbers.

For the complex number $z = a + ib$, a is called the *real part*, denoted by Re z and b is called the *imaginary part* denoted by Im z of the complex number z. For example, if $z = 2 + i5$, then Re $z = 2$ and Im $z = 5$.

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal if $a = c$ and $b = d$.

Example 1 If $4x + i(3x - y) = 3 + i(-6)$, where x and y are real numbers, then find the values of x and y.

Solution We have
$$4x + i(3x - y) = 3 + i(-6) \qquad \text{... (1)}$$
Equating the real and the imaginary parts of (1), we get
$$4x = 3, \ 3x - y = -6,$$

which, on solving simultaneously, give $x = \dfrac{3}{4}$ and $y = \dfrac{33}{4}$.

4.3 Algebra of Complex Numbers

In this Section, we shall develop the algebra of complex numbers.

4.3.1 *Addition of two complex numbers* Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the sum $z_1 + z_2$ is defined as follows:

$z_1 + z_2 = (a + c) + i(b + d)$, which is again a complex number.

For example, $(2 + i3) + (-6 + i5) = (2 - 6) + i(3 + 5) = -4 + i8$

The addition of complex numbers satisfy the following properties:

   (i) *The closure law* The sum of two complex numbers is a complex number, i.e., $z_1 + z_2$ is a complex number for all complex numbers z_1 and z_2.
  (ii) *The commutative law* For any two complex numbers z_1 and z_2, $z_1 + z_2 = z_2 + z_1$
 (iii) *The associative law* For any three complex numbers z_1, z_2, z_3, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
  (iv) *The existence of additive identity* There exists the complex number $0 + i\,0$ (denoted as 0), called the *additive identity* or the *zero complex number,* such that, for every complex number z, $z + 0 = z$.
   (v) *The existence of additive inverse* To every complex number $z = a + ib$, we have the complex number $-a + i(-b)$ (denoted as $-z$), called the *additive inverse* or *negative of z.* We observe that $z + (-z) = 0$ (the additive identity).

4.3.2 *Difference of two complex numbers* Given any two complex numbers z_1 and z_2, the difference $z_1 - z_2$ is defined as follows:
$$z_1 - z_2 = z_1 + (-z_2).$$
For example, $\qquad (6 + 3i) - (2 - i) = (6 + 3i) + (-2 + i) = 4 + 4i$

and $\qquad\qquad\qquad (2 - i) - (6 + 3i) = (2 - i) + (-6 - 3i) = -4 - 4i$

4.3.3 *Multiplication of two complex numbers* Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers. Then, the product $z_1 z_2$ is defined as follows:

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$

For example, $(3 + i5) (2 + i6) = (3 \times 2 - 5 \times 6) + i(3 \times 6 + 5 \times 2) = -24 + i28$

The multiplication of complex numbers possesses the following properties, which we state without proofs.

(i) **The closure law** The product of two complex numbers is a complex number, the product $z_1 z_2$ is a complex number for all complex numbers z_1 and z_2.

(ii) **The commutative law** For any two complex numbers z_1 and z_2,
$$z_1 z_2 = z_2 z_1.$$

(iii) **The associative law** For any three complex numbers z_1, z_2, z_3,
$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

(iv) **The existence of multiplicative identity** There exists the complex number $1 + i 0$ (denoted as 1), called the *multiplicative identity* such that $z.1 = z$, for every complex number z.

(v) **The existence of multiplicative inverse** For every non-zero complex number $z = a + ib$ or $a + bi(a \neq 0, b \neq 0)$, we have the complex number

$\dfrac{a}{a^2 + b^2} + i \dfrac{-b}{a^2 + b^2}$ (denoted by $\dfrac{1}{z}$ or z^{-1}), called the *multiplicative inverse* of z such that

$z.\dfrac{1}{z} = 1$ (the multiplicative identity).

(vi) **The distributive law** For any three complex numbers z_1, z_2, z_3,
(a) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$
(b) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$

4.3.4 *Division of two complex numbers* Given any two complex numbers z_1 and z_2, where $z_2 \neq 0$, the quotient $\dfrac{z_1}{z_2}$ is defined by

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2}$$

For example, let $z_1 = 6 + 3i$ and $z_2 = 2 - i$

Then $\dfrac{z_1}{z_2} = \left((6 + 3i) \times \dfrac{1}{2 - i} \right) = (6 + 3i) \left(\dfrac{2}{2^2 + (-1)^2} + i \dfrac{-(-1)}{2^2 + (-1)^2} \right)$

$$= (6 + 3i)\left(\frac{2+i}{5}\right) = \frac{1}{5}\left[12 - 3 + i(6 + 6)\right] = \frac{1}{5}(9 + 12i)$$

4.3.5 *Power of i* we know that

$$i^3 = i^2 i = (-1)\, i = -i\,, \qquad i^4 = \left(i^2\right)^2 = (-1)^2 = 1$$

$$i^5 = \left(i^2\right)^2 i = (-1)^2 i = i\,, \qquad i^6 = \left(i^2\right)^3 = (-1)^3 = -1\,, \text{ etc.}$$

Also, we have $\quad i^{-1} = \dfrac{1}{i} \times \dfrac{i}{i} = \dfrac{i}{-1} = -i, \quad i^{-2} = \dfrac{1}{i^2} = \dfrac{1}{-1} = -1,$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} \times \frac{i}{i} = \frac{i}{1} = i, \quad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

In general, for any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$

4.3.6 *The square roots of a negative real number*

Note that $i^2 = -1$ and $(-i)^2 = i^2 = -1$

Therefore, the square roots of -1 are $i, -i$. However, by the symbol $\sqrt{-1}$, we would mean i only.

Now, we can see that i and $-i$ both are the solutions of the equation $x^2 + 1 = 0$ or $x^2 = -1$.

Similarly $\qquad \left(\sqrt{3}\,i\right)^2 = \left(\sqrt{3}\right)^2 i^2 = 3\,(-1) = -3$

$$\left(-\sqrt{3}\,i\right)^2 = \left(-\sqrt{3}\right)^2 i^2 = -3$$

Therefore, the square roots of -3 are $\sqrt{3}\ i$ and $-\sqrt{3}i$.

Again, the symbol $\sqrt{-3}$ is meant to represent $\sqrt{3}\,i$ only, i.e., $\sqrt{-3} = \sqrt{3}\,i$.

Generally, if a is a positive real number, $\sqrt{-a} = \sqrt{a}\,\sqrt{-1} = \sqrt{a}\,i$,

We already know that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all positive real number a and b. This result also holds true when either $a > 0,\ b < 0$ or $a < 0,\ b > 0$. What if $a < 0,\ b < 0$? Let us examine.

Note that

$i^2 = \sqrt{-1}\,\sqrt{-1} = \sqrt{(-1)\,(-1)}$ (by assuming $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all real numbers)

$= \sqrt{1} = 1$, which is a contradiction to the fact that $i^2 = -1$.

Therefore, $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$ if both a and b are negative real numbers.

Further, if any of a and b is zero, then, clearly, $\sqrt{a} \times \sqrt{b} = \sqrt{ab} = 0$.

4.3.7 *Identities* We prove the following identity

$\left(z_1 + z_2\right)^2 = z_1^2 + z_2^2 + 2z_1z_2$, for all complex numbers z_1 and z_2.

Proof We have, $(z_1 + z_2)^2 = (z_1 + z_2)\,(z_1 + z_2)$,

$\qquad\qquad = (z_1 + z_2)\,z_1 + (z_1 + z_2)\,z_2$ \qquad (Distributive law)

$\qquad\qquad = z_1^2 + z_2z_1 + z_1z_2 + z_2^2$ \qquad (Distributive law)

$\qquad\qquad = z_1^2 + z_1z_2 + z_1z_2 + z_2^2$ \qquad (Commutative law of multiplication)

$\qquad\qquad = z_1^2 + 2z_1z_2 + z_2^2$

Similarly, we can prove the following identities:

(i) $\left(z_1 - z_2\right)^2 = z_1^2 - 2z_1z_2 + z_2^2$

(ii) $\left(z_1 + z_2\right)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$

(iii) $\left(z_1 - z_2\right)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$

(iv) $z_1^2 - z_2^2 = \left(z_1 + z_2\right)\left(z_1 - z_2\right)$

In fact, many other identities which are true for all real numbers, can be proved to be true for all complex numbers.

Example 2 Express the following in the form of $a + bi$:

(i) $(-5i)\left(\dfrac{1}{8}i\right)$ \hspace{3cm} (ii) $(-i)\,(2i)\left(-\dfrac{1}{8}i\right)^3$

Solution (i) $(-5i)\left(\dfrac{1}{8}i\right) = \dfrac{-5}{8}i^2 = \dfrac{-5}{8}(-1) = \dfrac{5}{8} = \dfrac{5}{8} + i0$

(ii) $(-i)\,(2i)\left(-\dfrac{1}{8}i\right)^3 = 2 \times \dfrac{1}{8 \times 8 \times 8} \times i^5 = \dfrac{1}{256}\left(i^2\right)^2\,i = \dfrac{1}{256}i$.

Example 3 Express $(5 - 3i)^3$ in the form $a + ib$.

Solution We have, $(5 - 3i)^3 = 5^3 - 3 \times 5^2 \times (3i) + 3 \times 5 (3i)^2 - (3i)^3$

$$= 125 - 225i - 135 + 27i = -10 - 198i.$$

Example 4 Express $\left(-\sqrt{3} + \sqrt{-2}\right)\left(2\sqrt{3} - i\right)$ in the form of $a + ib$

Solution We have, $\left(-\sqrt{3} + \sqrt{-2}\right)\left(2\sqrt{3} - i\right) = \left(-\sqrt{3} + \sqrt{2}\,i\right)\left(2\sqrt{3} - i\right)$

$$= -6 + \sqrt{3}i + 2\sqrt{6}i - \sqrt{2}\,i^2 = \left(-6 + \sqrt{2}\right) + \sqrt{3}\left(1 + 2\sqrt{2}\right)i$$

4.4 The Modulus and the Conjugate of a Complex Number

Let $z = a + ib$ be a complex number. Then, the modulus of z, denoted by $|z|$, is defined to be the non-negative real number $\sqrt{a^2 + b^2}$, i.e., $|z| = \sqrt{a^2 + b^2}$ and the conjugate of z, denoted as $\bar{z}$, is the complex number $a - ib$, i.e., $\bar{z} = a - ib$.

For example, $|3 + i| = \sqrt{3^2 + 1^2} = \sqrt{10}$, $|2 - 5i| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$,

and $\overline{3 + i} = 3 - i$, $\overline{2 - 5\,i} = 2 + 5\,i$, $\overline{-3i - 5} = 3i - 5$

Observe that the multiplicative inverse of the non-zero complex number z is given by

$$z^{-1} = \frac{1}{a + ib} = \frac{a}{a^2 + b^2} + i\frac{-b}{a^2 + b^2} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

or $z\,\bar{z} = |z|^2$

Furthermore, the following results can easily be derived.

For any two compex numbers z_1 and z_2, we have

(i) $|z_1 z_2| = |z_1||z_2|$ (ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ provided $|z_2| \neq 0$

(iii) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$ (iv) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ (v) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ provided $z_2 \neq 0$.

Example 5 Find the multiplicative inverse of $2 - 3i$.

Solution   Let $z = 2 - 3i$

Then $\bar{z} = 2 + 3i$ and $|z|^2 = 2^2 + (-3)^2 = 13$

Therefore,   the multiplicative inverse of $2 - 3i$ is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{2 + 3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

The above working can be reproduced in the following manner also,

$$z^{-1} = \frac{1}{2 - 3i} = \frac{2 + 3i}{(2 - 3i)(2 + 3i)}$$

$$= \frac{2 + 3i}{2^2 - (3i)^2} = \frac{2 + 3i}{13} = \frac{2}{13} + \frac{3}{13}i$$

Example 6 Express the following in the form $a + ib$

(i) $\dfrac{5 + \sqrt{2}i}{1 - \sqrt{2}i}$ (ii) i^{-35}

Solution   (i) We have, $\dfrac{5 + \sqrt{2}i}{1 - \sqrt{2}i} = \dfrac{5 + \sqrt{2}i}{1 - \sqrt{2}i} \times \dfrac{1 + \sqrt{2}i}{1 + \sqrt{2}i} = \dfrac{5 + 5\sqrt{2}i + \sqrt{2}i - 2}{1 - \left(\sqrt{2}i\right)^2}$

$$= \frac{3 + 6\sqrt{2}i}{1 + 2} = \frac{3(1 + 2\sqrt{2}i)}{3} = 1 + 2\sqrt{2}i$$

(ii) $i^{-35} = \dfrac{1}{i^{35}} = \dfrac{1}{\left(i^2\right)^{17} i} = \dfrac{1}{-i} \times \dfrac{i}{i} = \dfrac{i}{-i^2} = i$

EXERCISE 4.1

Express each of the complex number given in the Exercises 1 to 10 in the form $a + ib$.

1. $(5i)\left(-\dfrac{3}{5}i\right)$ **2.** $i^9 + i^{19}$ **3.** i^{-39}

4. $3(7 + i7) + i (7 + i7)$ **5.** $(1 - i) - (-1 + i6)$

6. $\left(\dfrac{1}{5} + i\dfrac{2}{5}\right) - \left(4 + i\dfrac{5}{2}\right)$ **7.** $\left[\left(\dfrac{1}{3} + i\dfrac{7}{3}\right) + \left(4 + i\dfrac{1}{3}\right)\right] - \left(-\dfrac{4}{3} + i\right)$

8. $(1 - i)^4$ **9.** $\left(\dfrac{1}{3} + 3i\right)^3$ **10.** $\left(-2 - \dfrac{1}{3}i\right)^3$

Find the multiplicative inverse of each of the complex numbers given in the Exercises 11 to 13.

11. $4 - 3i$ **12.** $\sqrt{5} + 3i$ **13.** $- i$

14. Express the following expression in the form of $a + ib$:

$$\dfrac{\left(3 + i\sqrt{5}\right)\left(3 - i\sqrt{5}\right)}{\left(\sqrt{3} + \sqrt{2}\,i\right) - \left(\sqrt{3} - i\sqrt{2}\right)}$$

4.5 Argand Plane and Polar Representation

We already know that corresponding to each ordered pair of real numbers (x, y), we get a unique point in the XY-plane and vice-versa with reference to a set of mutually perpendicular lines known as the x-axis and the y-axis. The complex number $x + iy$ which corresponds to the ordered pair (x, y) can be represented geometrically as the unique point P(x, y) in the XY-plane and vice-versa.

Some complex numbers such as $2 + 4i, - 2 + 3i, 0 + 1i, 2 + 0i, - 5 - 2i$ and $1 - 2i$ which correspond to the ordered

Fig 4.1

pairs $(2, 4)$, $(- 2, 3)$, $(0, 1)$, $(2, 0)$, $(-5, -2)$, and $(1, - 2)$, respectively, have been represented geometrically by the points A, B, C, D, E, and F, respectively in the Fig 4.1.

The plane having a complex number assigned to each of its point is called the *complex plane* or the *Argand plane*.
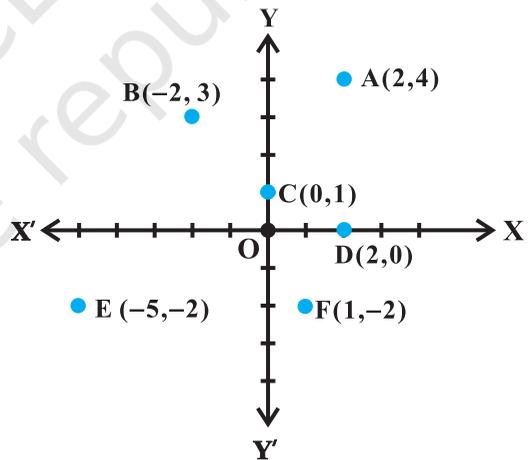
Obviously, in the Argand plane, the modulus of the complex number

$x + iy = \sqrt{x^2 + y^2}$ is the distance between the point P(x, y) and the origin O (0, 0) (Fig 4.2). The points on the x-axis corresponds to the complex numbers of the form $a + i\,0$ and the points on the y-axis corresponds to the complex numbers of the form
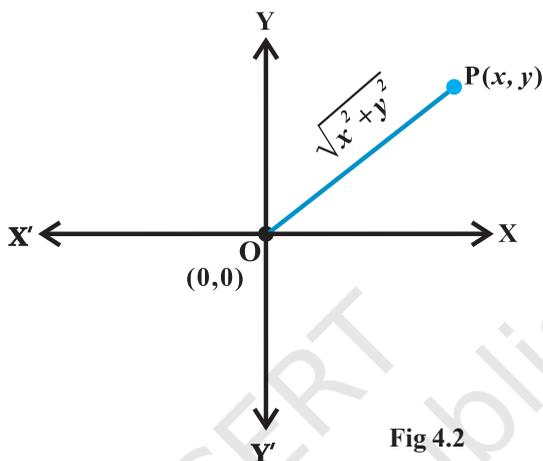

Fig 4.2

$0 + i\,b$. The x-axis and y-axis in the Argand plane are called, respectively, the *real axis* and the *imaginary axis*.

The representation of a complex number $z = x + iy$ and its conjugate $z = x - iy$ in the Argand plane are, respectively, the points P (x, y) and Q (x, $-y$).

Geometrically, the point (x, $-y$) is the mirror image of the point (x, y) on the real axis (Fig 4.3).
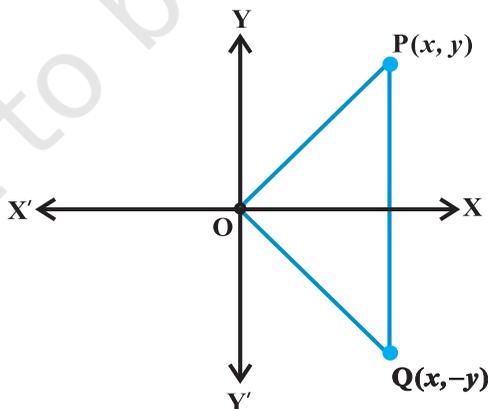

Fig 4.3

Miscellaneous Examples

Example 7 Find the conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.

Solution We have , $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

$$= \frac{6+9i-4i+6}{2-i+4i+2} \;=\; \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i}$$

$$= \frac{48-36i+20i+15}{16+9} = \frac{63-16i}{25} \;=\; \frac{63}{25} - \frac{16}{25}i$$

Therefore, conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ is $\dfrac{63}{25} + \dfrac{16}{25}i$.

Example 8 If $x + iy = \dfrac{a+ib}{a-ib}$, prove that $x^2 + y^2 = 1$.

Solution We have,

$$x + iy = \frac{(a+ib)(a+ib)}{(a-ib)(a+ib)} \;=\; \frac{a^2-b^2+2abi}{a^2+b^2} \;=\; \frac{a^2-b^2}{a^2+b^2} + \frac{2ab}{a^2+b^2}i$$

So that, $x - iy = \dfrac{a^2-b^2}{a^2+b^2} - \dfrac{2ab}{a^2+b^2}i$

Therefore,

$$x^2 + y^2 = (x+iy)\,(x-iy) = \frac{(a^2-b^2)^2}{(a^2+b^2)^2} + \frac{4a^2b^2}{(a^2+b^2)^2} \;=\; \frac{(a^2+b^2)^2}{(a^2+b^2)^2} \;=\; 1$$

Miscellaneous Exercise on Chapter 4

1. Evaluate: $\left[i^{18} + \left(\dfrac{1}{i} \right)^{25} \right]^3$.

2. For any two complex numbers z_1 and z_2, prove that
   $\mathrm{Re}\,(z_1\,z_2) = \mathrm{Re}\,z_1\,\mathrm{Re}\,z_2 - \mathrm{Im}z_1\,\mathrm{Im}z_2.$

3. Reduce $\left(\dfrac{1}{1-4i} - \dfrac{2}{1+i} \right)\left(\dfrac{3-4i}{5+i} \right)$ to the standard form .

4. If $x - iy = \sqrt{\dfrac{a-ib}{c-id}}$ prove that $\left(x^2 + y^2 \right)^2 = \dfrac{a^2 + b^2}{c^2 + d^2}$.

5. If $z_1 = 2 - i, z_2 = 1 + i$, find $\left| \dfrac{z_1 + z_2 + 1}{z_1 - z_2 + 1} \right|$.

6. If $a + ib = \dfrac{(x+i)^2}{2x^2 + 1}$, prove that $a^2 + b^2 = \dfrac{(x^2 + 1)^2}{\left(2x^2 + 1 \right)^2}$.

7. Let $z_1 = 2 - i, z_2 = -2 + i$. Find

(i) $\mathrm{Re}\left(\dfrac{z_1 z_2}{\overline{z}_1} \right)$,   (ii) $\mathrm{Im}\left(\dfrac{1}{z_1 \overline{z}_1} \right)$.

8. Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

9. Find the modulus of $\dfrac{1+i}{1-i} - \dfrac{1-i}{1+i}$.

10. If $(x + iy)^3 = u + iv$, then show that $\dfrac{u}{x} + \dfrac{v}{y} = 4(x^2 - y^2)$.

11. If α and β are different complex numbers with $|\beta| = 1$, then find $\left| \dfrac{\beta - \alpha}{1 - \overline{\alpha}\beta} \right|$.

12. Find the number of non-zero integral solutions of the equation $\left| 1 - i \right|^x = 2^x$.

13. If $(a + ib)(c + id)(e + if)(g + ih) = A + iB$, then show that
$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)(g^2 + h^2) = A^2 + B^2$

14. If $\left(\dfrac{1+i}{1-i} \right)^m = 1$, then find the least positive integral value of m.

Summary

◆ A number of the form $a + ib$, where a and b are real numbers, is called a *complex number*, a is called the *real part* and b is called the *imaginary* part of the complex number.

◆ Let $z_1 = a + ib$ and $z_2 = c + id$. Then
  (i) $z_1 + z_2 = (a + c) + i (b + d)$
  (ii) $z_1 z_2 = (ac - bd) + i (ad + bc)$

◆ For any non-zero complex number $z = a + ib$ ($a \neq 0, b \neq 0$), there exists the complex number $\dfrac{a}{a^2 + b^2} + i \dfrac{-b}{a^2 + b^2}$, denoted by $\dfrac{1}{z}$ or z^{-1}, called the *multiplicative inverse* of z such that $(a + ib) \dfrac{a}{a^2 + b^2} + i \dfrac{-b}{a^2 + b^2} = 1 + i0$ $=1$

◆ For any integer k, $i^{4k} = 1$, $i^{4k + 1} = i$, $i^{4k + 2} = -1$, $i^{4k + 3} = -i$

◆ The conjugate of the complex number $z = a + ib$, denoted by $\overline{z}$, is given by $\overline{z} = a - ib$.

Historical Note

The fact that square root of a negative number does not exist in the real number system was recognised by the Greeks. But the credit goes to the Indian mathematician *Mahavira* (850) who first stated this difficulty clearly. "He mentions in his work '*Ganitasara Sangraha*' as in the nature of things a negative (quantity) is not a square (quantity)', it has, therefore, no square root". *Bhaskara*, another Indian mathematician, also writes in his work *Bijaganita*, written in 1150. "There is no square root of a negative quantity, for it is not a square." *Cardan* (1545) considered the problem of solving

$$x + y = 10, \, xy = 40.$$

He obtained $x = 5 + \sqrt{-15}$ and $y = 5 - \sqrt{-15}$ as the solution of it, which was discarded by him by saying that these numbers are 'useless'. *Albert Girard* (about 1625) accepted square root of negative numbers and said that this will enable us to get as many roots as the degree of the polynomial equation. *Euler* was the first to introduce the symbol *i* for $\sqrt{-1}$ and *W.R. Hamilton* (about 1830) regarded the complex number $a + ib$ as an ordered pair of real numbers (a, b) thus giving it a purely mathematical definition and avoiding use of the so called '*imaginary numbers*'.

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